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THE p-VERSION OF THE FINITE ELEMENT METHOD FOR PARABOLIC EQUATIONS

PART I

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July 1988

INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

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The paper, which is the first in a series, presents some basic results of the p-version of the finite element method for parabolic equation.
The p-version is applied for the discretization in both variables t and x. It is assumed that in t only one element of degree $q \to \infty$ is used. Error estimates and numerical computations are presented.

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Abstract. The paper, which is the first in a series, presents some basic results of the p-version of the finite element method for parabolic equation.

The p-version is applied for the discretization in both variables t and x. It is assumed that in t only one element of degree $q \longrightarrow \infty$ is used. Error estimates and numerical computations are presented.

1. Introduction.

The finite element method has become the main tool in computational mechanics. The MAKABASE [1], [2] contains about 20,000 references on finite element and 2000 boundary element technology. Recently the new direction in the finite element theory and practice appeared, the p and h-p versions, which utilize high degree of elements. About 3 - 4 dozen references about p and h-p versions are available, all of them related to the elliptic problems. For the survey of today's state of the art, we refer to e.g. [3].

The present paper addresses the basic problems of the p-version for the parabolic equation with both variables, x and t discreted via p-version. It concentrates on the case when in the time variables only one interval is used. The paper gives the error estimates and presents some numerical aspects. ~~We~~ restrict themselves to the basic features of the method. Various generalizations will be presented in forthcoming papers.

2. The p-version for the initial value problem for an ordinary differential equation

2.1. Preliminaries and problem formulation.

Let $I = (-1,1)$, $\bar{I} = [-1,1]$, $t \in I$, $X = L_2(I)$ be the usual space furnished with the norm

$$\|u\|_X = \left[\int_{-1}^{1} u^2 dt \right]^{1/2}. \tag{2.1}$$

Let

$$\overset{\circ}{C} = \{v \in C^{\infty}(\bar{I}) \mid v(1) = 0\},$$

where $C^{\infty}(\bar{I})$ is the usual space of functions with all continuous derivatives on $\bar{I}$. For any $\lambda > 0$ and $v \in \mathring{C}^{1}$ we define

$$\|v\|_{Y_\lambda} = \|- \frac{\dot{v}}{\lambda} + \lambda v\|_X , \tag{2.2}$$

where we denoted $\dot{v} = \frac{dv}{dt}$. Let Y_λ be the completion of $\mathring{C}^{1}$ with respect to the norm $\|\cdot\|_{Y_\lambda}$.

<u>Lemma 2.1</u>. Let $v \in \mathring{C}^{1}$ and

$$\|v\|^2_{Z_\lambda} = \|\frac{\dot{v}}{\lambda}\|^2_X + \|\lambda v\|^2_X. \tag{2.3}$$

Then

$$\|v\|_{Z_\lambda} \le \|v\|_{Y_\lambda} \le \sqrt{2}\,\|v\|_{Z_\lambda}. \tag{2.4}$$

<u>Proof</u>. We have

$$\|v\|^2_{Y_\lambda} = \int_{-1}^{1} \left(- \frac{\dot{v}}{\lambda} + \lambda v\right)^2 dt = \int_{-1}^{1} \left[\left(\frac{\dot{v}}{\lambda}\right)^2 + (\lambda v)^2\right] dt - 2\int_{-1}^{1} \dot{v}v\,dt$$

$$= \|v\|^2_{Z_\lambda} - \int_{-1}^{1} \frac{d}{dt}(v^2(t))\,dt =$$

$$= \|v\|^2_{Z_\lambda} + v^2(-1) \ge \|v\|^2_{Z_\lambda}.$$

On the other hand

$$\|v\|^2_{Y_\lambda} \le 2\left(\|\frac{\dot{v}}{\lambda}\|^2_X + \|\lambda v\|^2_X\right) \le 2\|v\|^2_{Z_\lambda}.$$ ∎

Lemma 2.1 implies that the spaces Y_λ and Z_λ are equivalent.

On $X \times Y_\lambda$, $u \in X$, $v \in Y_\lambda$ we define the bilinear form

$$B_\lambda(u,v) = \int_{-1}^{1} u\left[-\frac{\dot v}{\lambda} + \lambda v\right]dt. \tag{2.5}$$

Further, let $F \in Y'_\lambda$ be a linear functional on Y_λ.

We will define now

<u>Problem</u> $\mathcal{P}_\lambda$. Given $F \in Y'_\lambda$, find $u_0 \in X$ such that

$$B_\lambda(u_0,v) = F(v) \quad \forall\, v \in Y_\lambda.$$

<u>Theorem 2.2</u>. Problem $\mathcal{P}_\lambda$ has a unique solution $u_0 \in X$ and $\|u_0\|_X \le \|F\|_{Y'_\lambda}$.

<u>Proof</u>. By Schwarz inequality

$$|B_\lambda(u,v)| \le \|u\|_X \|v\|_{Y_\lambda}. \tag{2.6a}$$

Given $u \in X$, then there is $v \in Y_\lambda$ such that

$$-\frac{\dot v}{\lambda} + \lambda v = u.$$

Obviously $\|v\|_{Y_\lambda} = \|u\|_X$. This yields immediately

$$\inf_{\substack{u\in X\\ \|u\|_X=1}} \ \sup_{\substack{v\in Y_\lambda\\ \|v\|_{Y_\lambda}\le 1}} |B_\lambda(u,v)| \ge 1 \tag{2.6b}$$

and analogously

$$\inf_{\substack{v\in Y_\lambda\\ \|v\|_{Y_\lambda}=1}} \ \sup_{\substack{u\in X\\ \|u\|_X\le 1}} |B_\lambda(u,v)| \ge 1. \tag{2.6c}$$

Now Theorem 2.2 follows immediately from Theorem 5.2.1 of [4]. ∎

Let us show that the solution of the problem $\mathcal{P}_\lambda$ is a weak solution u_0 of the initial value problem

(2.7a) $$\frac{\dot{u}_0}{\lambda} + \lambda u_0 = f$$

(2.7b) $$u_0(-1) = a\lambda$$

if

(2.7c) $$F(v) = \int_{-1}^{1} fv dt + av(-1).$$

To prove it, it suffices to show that if $f \in C^0(\bar{I})$ and $\bar{u}_0 \in C^1(\bar{I})$ solves (2.7a,b) then $\bar{u}_0$ is the solution of the problem $\mathcal{P}_\lambda$.

We indeed have for any $v \in \mathcal{E}^)$

$$B_\lambda(\bar{u}_0, v) = \int_{-1}^{1} \left[\frac{\dot{\bar{u}}_0}{\lambda} + \lambda\bar{u}_0\right] vdt + \bar{u}_0(-1)v(-1)\frac{1}{\lambda} = \int_{-1}^{1} fv\ dt + av(-1).$$

This shows that any classical solution (i.e., a solution belonging to $C^1(\bar{I})$) is also the solution of the problem $\mathcal{P}_\lambda$ with F given by (2.7c) and hence the problem $\mathcal{P}_\lambda$ is a weak formulation of (2.7a,b).

We mention that because of Lemma 2.1 and the Sobolev imbedding theorem $v \in Y_\lambda \hookrightarrow C^0(\bar{I})$ and hence $F(v) = \int_{-1}^{1} fvdt + av(-1) \in Y'_\lambda$. $F(v)$ can be obviously identified with a function from $H^{-1}(I)$. We note that $u \in X$ has in general no trace in $t = -1$, but we still have an initial value problem. The initial condition is now of the type of the natural boundary condition and not the essential one.

2.2. The p-version of the finite element method for the problem $\mathcal{P}_\lambda$.

Let $q \geq 1$, integer

(2.8) $$S = S^{q-1} = \{u \in X \mid u \text{ is a polynomial of degree } q-1\}$$

(2.9) $$V = \mathcal{S}^{)q} = \{v \in Y_\lambda \mid v \text{ is a polynomial of degree } q\}.$$

Theorem 2.3.

i) Let $u \in S$, $v \in V$, then

$$|B_\lambda(u,v)| \le \|u\|_X \|v\|_{Y_\lambda}. \tag{2.10}$$

ii)

$$d_\lambda(q) = \inf_{\substack{v \in \mathring{S}^q \\ \|v\|_{Y_\lambda}=1}} \sup_{\substack{u \in S^{q-1} \\ \|u\|_X \le 1}} |B_\lambda(u,v)| \ge \frac{1}{2} q^{-1/2}. \tag{2.11}$$

iii) Let $u \in S$, $u \ne 0$, then

$$\sup_{\substack{v \in \mathring{S}^q \\ \|v\|_{Y_\lambda}=1}} |B_\lambda(u,v)| > 0. \tag{2.12}$$

Proof.

i) (2.10) follows from (2.6a).

ii) Let P_k, $k = 0,1,\ldots$ be the Legendre polynomials. Then $v \in V$ can be written in the form

$$v = \sum_{k=0}^{q} \beta_k P_k \tag{2.13a}$$

with the constraint

$$0 = v(1) = \sum_{k=0}^{q} \beta_k. \tag{2.13b}$$

Let π_{q-1} be the X-orthogonal projection operator of X onto S and let $z = \pi_{q-1} w$,

$$w = -\frac{\dot{v}}{\lambda} + \lambda v, \quad v \in V.$$

Then for

$$z = -\frac{\dot{v}}{\lambda} + \lambda z_1 \quad \text{and} \quad z_1 = \pi_{q-1} v,$$

we have

$$z \in S \quad \text{and} \quad z_1 = \sum_{k=0}^{q-1} \beta_k P_k.$$

Then

$$B_\lambda(z, v) = \int_{-1}^{1} \left(-\frac{\dot{v}}{\lambda} + \lambda z_1\right)\left(-\frac{\dot{v}}{\lambda} + \lambda v\right) dt$$

$$= \|\frac{\dot{v}}{\lambda}\|_X^2 - \int_{-1}^{1} z_1 \dot{v} dt - \int_{-1}^{1} \dot{v} v dt + \lambda^2 \int_{-1}^{1} z_1 v dt = \|z\|_X^2.$$

Because

$$\int_{-1}^{1} z_1 \dot{v} dt = \int_{-1}^{1} \dot{v} v dt$$

and

$$2\int_{-1}^{1} v\dot{v} dt = -v^2(-1)$$

we see that

$$B_\lambda(z, v) \geq \|\frac{\dot{v}}{\lambda}\|_X^2 + \lambda^2 \int_{-1}^{1} z_1 v dt.$$

Further $v(1) = 0$ and therefore from (2.13b)

$$-\beta_q = \sum_{k=0}^{q-1} \beta_k$$

and

$$|\beta_q|^2 \leq \left(\sum_{k=0}^{q-1} \frac{2\beta_k^2}{2k+1}\right)\left(\sum_{k=0}^{q-1} \frac{(2k+1)}{2}\right) = \frac{q^2}{2} \sum_{k=0}^{q-1} \frac{2\beta_k^2}{2k+1}.$$

Hence

$$\|v\|_X^2 = \sum_{k=0}^{q} \frac{2\beta_k^2}{2k+1} \le \left[1 + \frac{q^2}{2(2q+1)}\right] \sum_{k=0}^{q-1} \frac{2\beta_k^2}{2k+1} \le 2q \sum_{k=0}^{q-1} \frac{2\beta_k^2}{2k+1}$$

and therefore

$$B_\lambda(z,v) = \|\tfrac{\dot v}{\lambda}\|_X^2 + \lambda^2\int_{-1}^{1} z_1 v\,dt = \|\tfrac{\dot v}{\lambda}\|_X^2 + \lambda^2 \sum_{k=0}^{q-1} \frac{2\beta_k^2}{2k+1} \ge \|\tfrac{\dot v}{\lambda}\|_X^2 + \frac{1}{2q}\|\lambda v\|_X^2$$

$$\ge \frac{1}{4q}\|v\|_{Y_\lambda}^2 \quad \text{(by Lemma 2.1).}$$

Realizing that $\|z\|_X^2 = B_\lambda(z,v)$, (2.11) follows.

iii) Given $u \in S^{q-1}$, let $v(t) = \int_t^1 u\,d\bar t$. Then $v \in \mathring{S}^q$ and

(2.14) $$B_\lambda(u,v) = \int_{-1}^{1} \left[\frac{u^2}{\lambda} - \lambda \dot v v\right] dt \ge \frac{1}{\lambda}\|u\|_X^2 > 0,$$

and (2.12) follows. ∎

We can now define the p-version of the finite element method for the problem $\mathcal{P}_\lambda$: Given $F \in Y'_\lambda$ and $q \ge 1$, integer, find $u_q \in S^{q-1}$ such that

(2.15) $$B_\lambda(u_q, v) = F(v) \quad \forall\, v \in \mathring{S}^q = V.$$

Theorem 2.3, together with Theorem 6.2.1 of [4] yields

<u>Theorem 2.4</u>. There is a unique u_q satisfying (2.15). If $u_0 \in X$ is the exact solution of the problem $\mathcal{P}_\lambda$, then

(2.16) $$\|u_0 - u_q\|_X \le [1+2q^{1/2}] \inf_{w\in S} \|u_0 - w\|_X.$$ ∎

The p-version of the finite element method reduces to the solution of a system of linear equations. Let $P_k(t)$ be the Legendre polynomials, then we can write

$$u_q = \sum_{k=0}^{q-1} \alpha_k P_k$$

and

$$v = \sum_{k=0}^{q-1} \beta_k (P_{k+1} - P_k) \in V.$$

Hence (2.15) reduces to a system of linear equations for α_k:

$$\sum_{k=0}^{q-1} \alpha_k B_\lambda(P_k, P_{j+1} - P_j) = F(P_{j+1} - P_j), \quad j = 0, \ldots, q-1.$$

Of course, we can use another basis function of S and V, too.

If (2.11) is optimal, i.e., $d(q) \le Cq^{-1/2}$, then (see [5]) there exists $\bar{u} \in X$, which is solution of the problem $\mathcal{P}_\lambda$ for certain $F \in Y'_\lambda$, such that the finite element solution diverges, i.e.

$$\|\bar{u} - u_q\|_X \longrightarrow \infty \quad \text{as} \quad q \longrightarrow \infty.$$

On the other hand, for a restrictive class γ of the solutions u of the problem $\mathcal{P}_\lambda$, we have

$$\|u - u_q\|_X \le C \inf_{w \in S} \|u - w\|_X$$

with C independent of q or growing with q much more slowly than $q^{1/2}$. In the next sections we will address more of these aspects.

2.3. <u>The optimality of (2.11)</u>.

Let

$$u = \sum_{k=0}^{q-1} \alpha_k P_k$$

$$v = \sum_{k=0}^{q} \beta_k P_k, \quad \beta_q = -\sum_{k=0}^{q-1} \beta_k.$$

Then

$$\|u\|_X^2 = \sum_{k=0}^{q-1} \frac{2\alpha_k^2}{2k+1}$$

$$\|v\|_{Y_\lambda}^2 = \frac{\sum_{j=0}^{q} \sum_{k=0}^{q} \beta_j \beta_k \int_{-1}^{1} \dot{P}_j \dot{P}_k dt}{\lambda^2} + \lambda^2 \left[\sum_{k=0}^{q-1} \frac{2\beta_k^2}{2k+1} + \frac{2}{2q+1} \left[\sum_{k=0}^{q-1} \beta_k \right]^2 \right]$$

and

(2.17)
$$B_\lambda(u,v) = \frac{-\sum_{j=0}^{q} \sum_{k=0}^{q-1} \alpha_k \beta_j \int_{-1}^{1} \dot{P}_j P_k dt}{\lambda} + \lambda \sum_{j=0}^{q-1} \frac{2\alpha_j \beta_j}{2j+1}.$$

Obviously

$$R = \lambda \sup_{\substack{\alpha_j \\ \sum_{j=0}^{q-1} 2\alpha_j^2/(2j+1)=1}} \left| \sum_{j=0}^{q-1} \frac{2\alpha_j \beta_j}{2j+1} \right| = \lambda \left[\sum_{j=0}^{q-1} \beta_j^2 \frac{2}{(2j+1)} \right]^{1/2}.$$

Let

(2.18)
$$\inf \lambda^2 \sum_{j=0}^{q-1} \beta_j^2 \frac{2}{(2j+1)} = Q$$

where inf is taken over β_j such that

(2.19)
$$\lambda^2 \left[\sum_{j=0}^{q-1} \frac{2\beta_j^2}{2j+1} + \frac{2}{2q+1} \left[\sum_{j=0}^{q-1} \beta_j \right]^2 \right] = 1.$$

Select

(2.20)
$$\beta_j = \frac{K}{\lambda}(2j+1) \quad j = 0, \ldots, q-1$$

with K determined so that (2.19) holds. Then

$$C_1 q^3 \le K^{-2} \le C_2 q^3,$$

with C_1, C_2 independent of q and

$$Q \le Cq^{-1},$$

where C is independent of q. Assuming that (2.19) holds, it is obvious that there is a function $0 < \mathcal{H}_1(x) < \infty$, $0 < x < \infty$ such that

$$\|v\|^2_{Y_\lambda} \le \mathcal{H}_1(q)\frac{1}{\lambda^2} + 1.$$

Further there is a function $0 < \mathcal{H}_2(x) < \infty$, $0 < x < \infty$ such that

$$\sup_{\substack{\alpha_j \\ \sum_{j=0}^{q-1} 2\alpha_j^2/(2j+1)=1}} |B_\lambda(u,v)| \le \mathcal{H}_2(q)\frac{1}{\lambda} + R.$$

Hence

$$d_\lambda(q) = \inf_{\|v\|_{Y_\lambda}=1} \sup_{\|u\|_X \le 1} |B_\lambda(u,v)| \le (\mathcal{H}_2(q)\frac{1}{\lambda} + C^{1/2}q^{-1/2})(1 + \frac{\mathcal{H}_1(q)}{\lambda^2})^{-1/2}$$

and we have proved

<u>Theorem 2.5</u>. If $\lambda \ge C^{-1/2}q^{1/2}\mathcal{H}_2(q)$, $\lambda \ge \mathcal{H}_1^{1/2}(q)$, then

$$d_\lambda(q) \le 4C^{1/2}q^{-1/2}$$

where C is the same constant as in (2.20), independent of q. ■

We see that for λ which is large in comparison with q the estimate (2.11) is optimal, i.e.,

$$\limsup_{\lambda\to\infty} d_\lambda(q) \le Cq^{-1/2}.$$

It does not mean that for fixed λ and q, $d_\lambda(q)$ is small. In fact we can expect that $d_\lambda(q)$ achieves minimum for some finite q_0 depending on λ.

The constant $d_\lambda(q)$ can be computed numerically. Let $u \in S$, $v \in V$,

$$u(t) = \sum_{j=1}^{q} \alpha_j \hat{P}_{j-1}(t), \tag{2.21}$$

and

$$v(t) = \sum_{j=1}^{q} \beta_j \mu_j(t), \tag{2.22}$$

where $\mu_1 = \frac{1-t}{2}$, $\mu_j(t) = \int_t^1 \hat{P}_{j-1}(\eta)d\eta$, $j = 2,3,\ldots,q$, and $\hat{P}_j(t)$, $j = 0,1,\ldots$ are normalized Legendre polynomials of degree j. Then

$$B_\lambda(u,v) = \beta^T A\alpha,$$

where

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_2 & a_2 & b_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{q-1} \\ & 0 & & c_q & a_q \end{bmatrix} \tag{2.23}$$

with

$$a_1 = \frac{\lambda^2+1}{\sqrt{2}\lambda},$$

$$a_i = -\frac{1}{\lambda}, \quad i = 1,2,\ldots,q$$

$$b_1 = \frac{-\lambda}{\sqrt{6}},$$

$$b_i = \frac{\lambda}{\sqrt{(2i-1)(2i+1)}}, \quad i = 1,2,\ldots,q-1$$

$$c_i = \frac{-\lambda}{\sqrt{(2i-1)(2i+3)}}, \quad i = 2,\dots,q$$

$$\alpha = [\alpha_1,\dots,\alpha_q]^T$$

$$\beta = [\beta_1,\dots,\beta_q]^T$$

$$\|u\|_X^2 = \alpha^T\alpha$$

$$\|v\|_{Y_\lambda}^2 = \beta^T C \beta,$$

where

$$C = \begin{bmatrix} d_1 & e_1 & f_1 & & & & & 0 \\ e_1 & d_2 & e_2 & f_2 & & & & \\ f_1 & e_2 & d_3 & e_3 & f_3 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 0 & & f_{q-4} & e_{q-3} & d_{q-2} & e_{q-2} & f_{q-2} \\ & & & & f_{q-3} & e_{q-2} & d_{q-1} & e_{q-1} \\ & & & & & f_{q-2} & e_{q-1} & d_q \end{bmatrix}$$

$$d_1 = \frac{3+6\lambda^2+4\lambda^2}{6\lambda^2}$$

$$d_i = \frac{1}{\lambda} + \frac{2\lambda^3}{(2i+1)(2i-3)}, \quad i = 2,\dots,q$$

$$e_1 = -\frac{\lambda^2}{\sqrt{6}}$$

$$e_i = 0, \quad i=2,\dots,q-1$$

$$f_1 = \frac{\lambda^2}{3\sqrt{10}}$$

$$f_i = \frac{-\lambda^2}{\sqrt{(2i+3)(2i-1)(2i+1)}}, \quad i = 2,\dots,q-2.$$

Now,

$$d_\lambda(q) = \inf_{\beta^T C\beta \le 1} \frac{\beta^T AA^T \beta}{(\beta^T AA^T \beta)^{1/2}} = \inf_{\beta} \left(\frac{\beta^T AA^T \beta}{\beta^T C\beta}\right)^{1/2}$$

and hence $d_\lambda^2(q)$ is the smallest eigenvalue of the problem

$$AA^T \beta = d_\lambda^2(q) C\beta.$$

Table 2.1 presents the values of $d_\lambda(q)$ for various q and λ. Figures 2.1 and 2.2 present $d_\lambda(q)$ in the dependence on q and λ. We see that for any fixed λ, $d_\lambda(q)$ first decreases with q and then increases, for fixed q, $d_\lambda(q)$ decreases with λ and

$$\lim_{\lambda\to\infty} \frac{d_\lambda(q_1)}{d_\lambda(q_2)} = \left(\frac{q_1}{q_2}\right)^{1/2}.$$

The detailed theoretical analysis of the structure of $d_\lambda(q)$ is not available.

$q \backslash \lambda$	1	3	5	10	20	25	40	50	100
3	.98680	.79701	.71727	.67610	.66514	.66381	.66237	.66203	.66159
5	.99506	.82899	.67937	.58675	.56138	.55829	.55493	.55415	.55312
10	.99875	.92492	.74648	.51874	.44260	.43325	.42311	.42076	.41764
15	.99944	.96118	.82747	.53586	.39710	.37941	.36024	.35582	.34994
20	.99969	.97678	.87845	.58089	.38175	.35411	.32406	.31715	.30796
25	.99980	.98468	.91095	.62875	.38291	.34423	.30172	.29194	.27897
30	.99986	.98918	.93257	.67097	.39403	.34403	.28781	.27482	.25763
40	.99992	.99381	.95813	.73776	.43085	.36091	.27550	.25519	.22834
50	.99995	.99601	.97179	.78735	.47194	.38901	.27593	.24743	.20947
100	.99999	.99899	.99240	.91037	.62570	.52393	.34373	.28026	.17611

Table 2.1. The values of $d_\lambda(q)$.

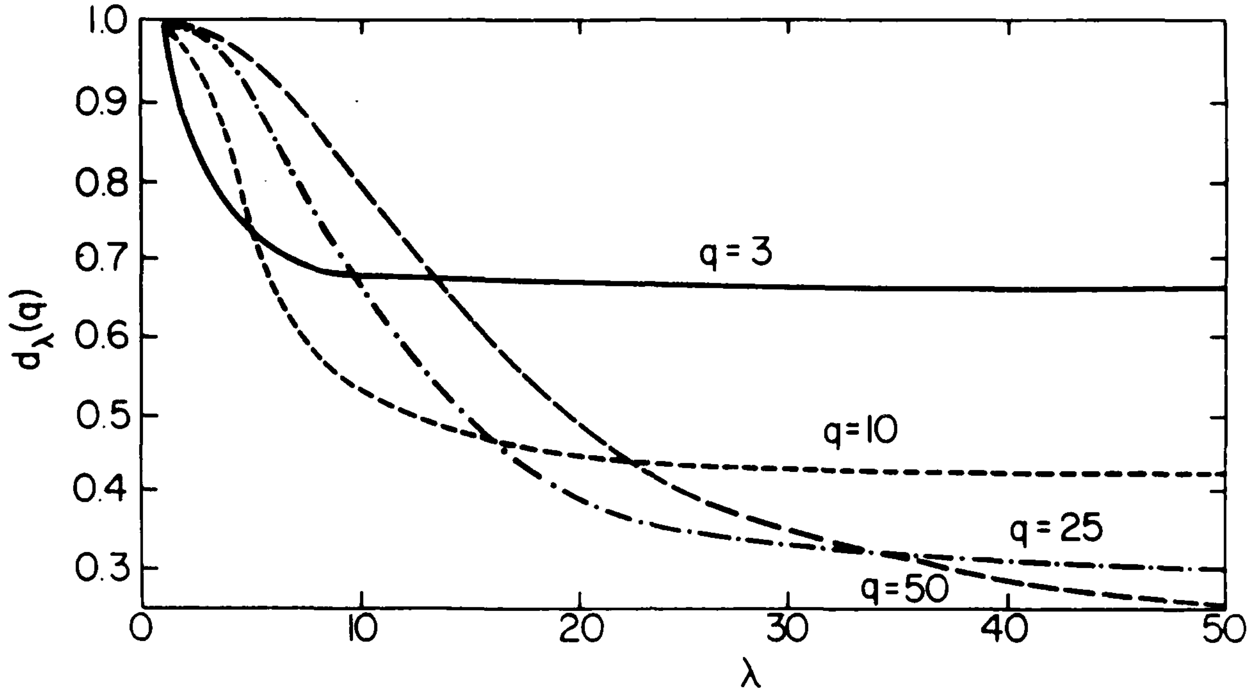

Figure 2.1. $d_\lambda(q)$ in dependence on λ with the polynomial degree q fixed.

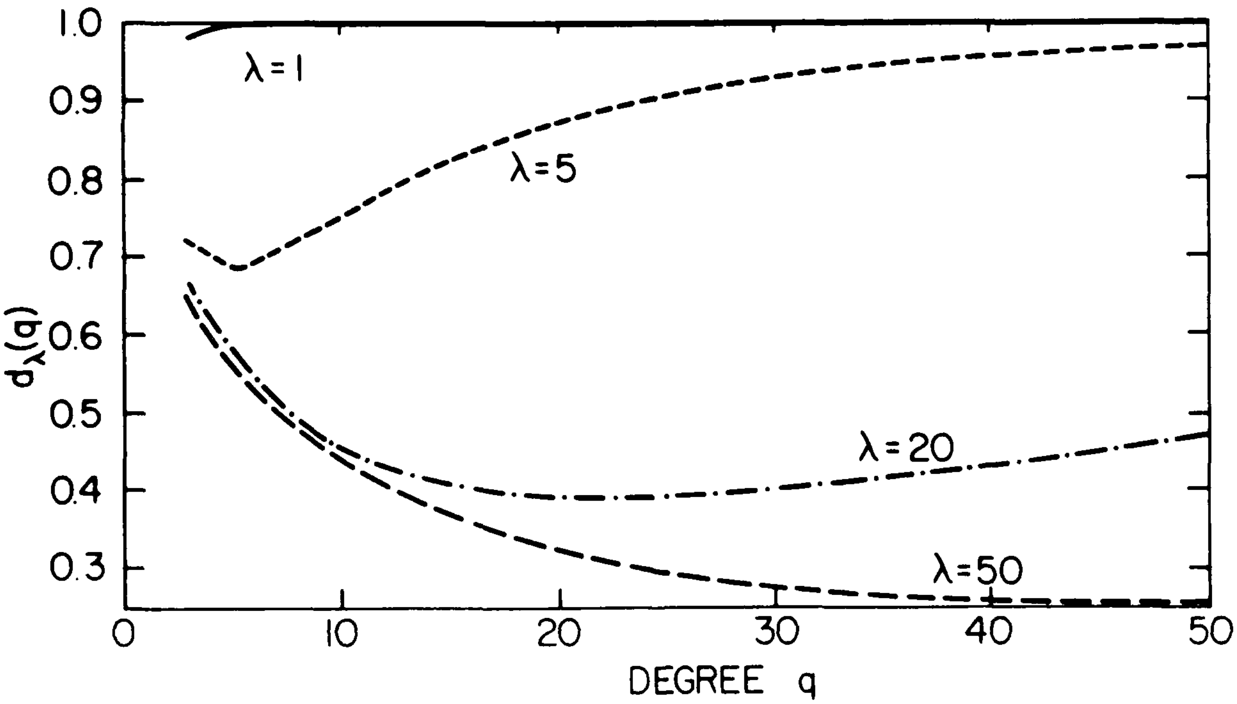

Figure 2.2. $d_\lambda(q)$ in dependence on the polynomial degree q with fixed λ.

2.4. The set of perfect solutions.

We have shown in section 2.2 and 2.3 that there are solutions $u \in X$ such that

$$\frac{\|u-u_q\|_X}{\inf\limits_{w\in S} \|u-w\|_X} = R_{q,\lambda}(u) \tag{2.25}$$

can be arbitrarily large provided that q and λ are sufficiently large. On the other hand, we have proven that

$$R_{q,\lambda}(u) \le Cq^{1/2}$$

where C is independent of q and u. Let $\varphi(x) < \infty$, $1 < x < \infty$ be a non-decreasing function such that $\frac{\varphi(x)}{x^{1/2}} \longrightarrow 0$ as $x \longrightarrow \infty$. Then if $D > 0$ is arbitrary, there exists $u \in X$ such that

$$\sup_{q,\lambda} R_{q,\lambda}(u)\varphi^{-1}(q) > D.$$

Hence we can ask to characterize the set $\gamma(\varphi) \subset X$ depending on φ such that

$$\sup_{u\in\gamma} \sup_{q,\lambda} R_{q,\lambda}(u)\varphi^{-1}(q) < \infty.$$

The set $\gamma(\varphi)$ will be called the set of φ-perfect solutions. An especially important case is $\gamma(\varphi)$ for $\varphi = 1$. It is not easy to give a precise characterization of the set $\gamma(\varphi)$ although its importance is obvious. We will give only some sufficient conditions and based upon numerical experiments we will also formulate a conjecture.

Let $u \in X$, $u = \sum_{j=0}^{\infty} \alpha_j \hat{P}_j$. Then

$$\inf_{w\in S} \|u - w\|_X = \|u - \bar{u}\|_X,$$

where

$$\bar{u} = \sum_{j=0}^{q-1} \alpha_j \hat{P}_j$$

and

$$\|u - \bar{u}\|_X^2 = \sum_{j=q}^{\infty} \alpha_j^2.$$

Let u_q be the finite element solution defined by (2.15) and

$$u - u_q = u - \bar{u} - z$$

with

$$z = u_q - \bar{u} \in S.$$

Then

$$B_\lambda(z,v) = B_\lambda(u_q - u + u - \bar{u}, v) = B_\lambda(u - \bar{u}, v) \quad \forall\ v \in V.$$

For $v = \sum_{k=0}^{q} \beta_k \hat{P}_k$

$$B_\lambda(z,v) = \int_{-1}^{1} (u - \bar{u}) \left[- \frac{\dot{v}}{\lambda} + \lambda v \right] dt = \lambda \beta_q \alpha_q.$$

Because (by Lemma 2.1)

$$\lambda^2 \beta_q^2 \le \lambda^2 \|v\|_X^2 \le \|v\|_{Y_\lambda}^2$$

we get

$$|B_\lambda(z,v)| \le |\alpha_q| \|v\|_{Y_\lambda}$$

and hence using (2.11)

$$\|z\|_X \le 2 |\alpha_q| q^{1/2}.$$

Thus

$$\|u-u_q\|_X \le \left(\left[\sum_{j=q}^{\infty}\alpha_j^2\right]^{1/2} + 2|\alpha_q|q^{1/2}\right) = \inf_{w\in S}\|u-w\|_X\left[1+2\,\frac{|\alpha_q|q^{1/2}}{\left(\sum_{j=q}^{\infty}\alpha_j^2\right)^{1/2}}\right]$$

which leads to the following

<u>Theorem 2.6</u>. Let $\varphi(x)$ be a nondecreasing function, $u \in X$, $u = \sum_{k=0}^{\infty}\alpha_k\hat{P}_k$ and

$$\sup_q \left[1+2\,\frac{|\alpha_q|q^{1/2}}{\left(\sum_{j=q}^{\infty}\alpha_j^2\right)^{1/2}}\right]\varphi^{-1}(q) < \infty. \tag{2.26}$$

Then

$$u \in \gamma(\varphi). \qquad \blacksquare$$

Consider some concrete examples:

i) $$u_1 = \sqrt{1-t^2} = \frac{4}{3}P_0(t) - 4\sum_{k=1}^{\infty}\frac{\hat{P}_k(t)}{\sqrt{2k+1}(2k-1)(2k+3)}$$

and hence u_1 is a φ-perfect solution for $\varphi = 1$.

ii) Let $0 < \rho < 1$ then

$$u_2 = \frac{1}{\sqrt{1-2x\rho+\rho^2}} = \sum_{k=0}^{\infty}\hat{P}_k\,\frac{\sqrt{2}}{\sqrt{2k+1}}\,\rho^k.$$

In this case Theorem 2.6 gives no indication whether u_2 is a φ-perfect solution for any φ (with exception of course of $\varphi = x^{1/2}$).

The observation about perfect solutions is practically important. Theorem 2.6 shows that if the solution is unsmooth, the finite element solution has essentially the same accuracy as the best approximation also for large q which is needed to get an acceptable accuracy. If the solution is smooth then the acceptable accuracy is achieved for small q and hence the factor $q^{1/2}$ is not important.

2.5. A numerical example.

Let us consider the problem

$$\frac{\dot{u}}{\lambda} + \lambda u = \lambda \tag{2.27a}$$

$$u(-1) = 0. \tag{2.27b}$$

The solution u is

$$u(t) = 1 - e^{-\lambda^2(t+1)}.$$

Assume that the basis functions of S and V are given by (2.21) and (2.22), respectively. Then the p-version reduces to the solution of linear system

$$A\alpha = F$$

where A is given by (2.23) and

$$F = [\lambda, -\frac{2\lambda}{\sqrt{6}}, 0 \ldots 0]^T.$$

The values of the function $R_{q,\lambda}(u)$ given by (2.25) are given in the Table 2.2. Computations were performed in double precision. Figure 2.3 presents $R_{q,\lambda}(u)$ in dependence on the polynomial degree q for various λ. The shown slope is the theoretical one based on Theorem 2.4 $(\mu = 0.5)$.

q \ λ	1	3	5	10	20	25	40	50	100
3	1.01317	1.14719	1.11079	1.03955	1.01089	1.00705	1.00279	1.00179	1.00045
5	1.00493	1.14853	1.19126	1.09510	1.02909	1.01910	1.00767	1.00494	1.00125
10	1.00000	1.07101	1.20361	1.23124	1.10008	1.06915	1.02947	1.01926	1.00495
15	1.00000	1.03765	1.14908	1.28929	1.18268	1.13488	1.06241	1.04165	1.01103
20	1.00000	1.00018	1.10947	1.28816	1.25360	1.20130	1.10258	1.07027	1.01934
25	1.00000	1.00000	1.08241	1.26821	1.30273	1.25826	1.14601	1.10308	1.02970
30	1.00000	1.00000	1.02000	1.24639	1.33032	1.30113	1.18923	1.13803	1.04190
40	1.00000	1.00000	1.00000	1.20763	1.34308	1.34560	1.26537	1.20737	1.07083
50	1.00000	1.00000	1.00000	1.17530	1.33333	1.35521	1.32000	1.26778	1.10408
100	1.00000	1.00000	1.00000	1.00000	1.26922	1.31420	1.37144	1.38065	1.27263

Table 2.2. The values of $R_{q,\lambda}(u)$ for various q and λ.

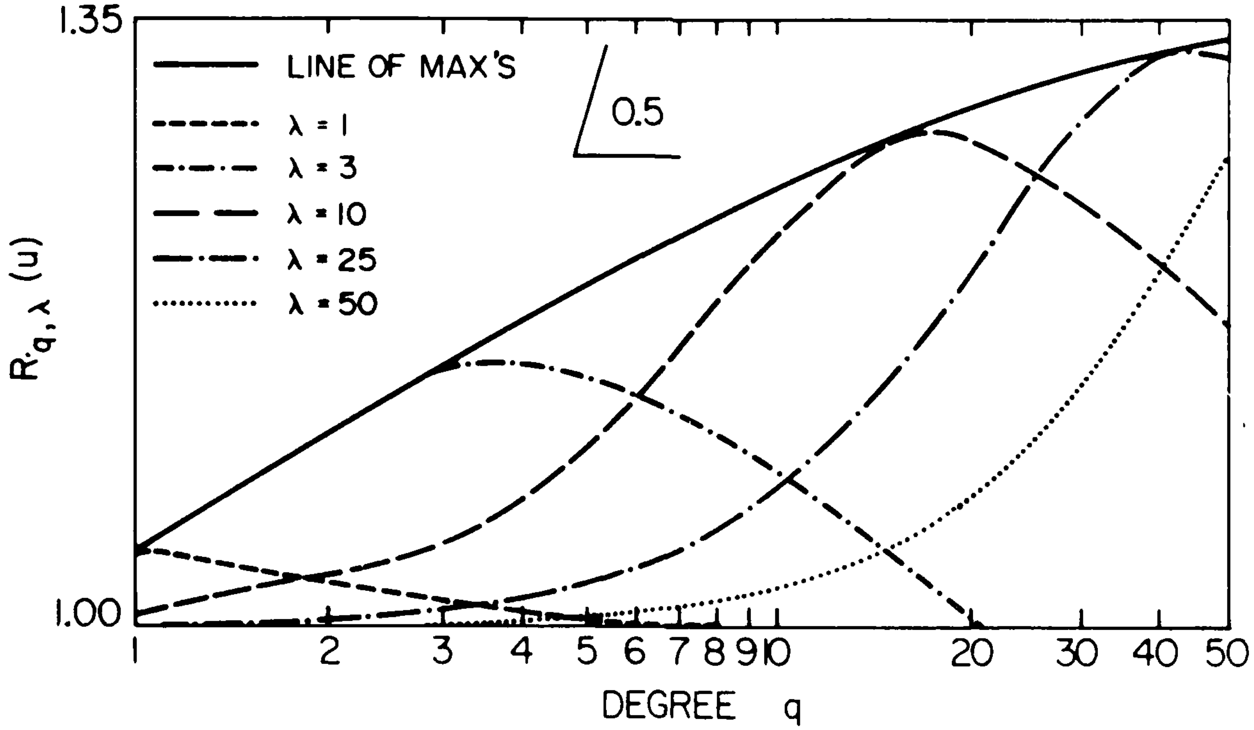

Figure 2.3. $R_{q,\lambda}(u)$ in dependence on the polynomial degree q, slope $\mu = 0.5$ based on (2.16).

Table 2.2 and Figure 2.3 show that for large λ, $R_{q,\lambda}(u)$ grows, but stays bounded (realize that we have used $q = 100$ leading to $q^{1/2} = 10$ and $d_{100}(100) = 0.17611$ in Table 2.1). It is interesting to observe that for fixed λ, $R_{q,\lambda}(u)$ first increases and then decreases to 1. Hence Table 2.2 and Figure 2.3 suggest

Conjecture A. Let $u_0 = e^{-\lambda^2(t+1)}$. Then $u_0 \in \gamma(\varphi)$ with $\varphi = 1$ or $\varphi(x) = \log(x)$. In fact we have seen that for any λ and any q in a practical range

$$R_{q,\lambda}(u) \le 1.5.$$

2.6. The error estimates.

We have shown in previous sections that

$$\|u - u_q\|_X \le C(q) \inf_{w \in S} \|u - w\|_X,$$

where $X = L_2(I)$. Assuming that

$$u = \sum_{k=0}^{\infty} \alpha_k \hat{P}_k$$

we have

$$\inf_{w \in S} \|u - w\|_X = \left[\sum_{k=q}^{\infty} \alpha_k^2\right]^{1/2}$$

and hence we deal with the error in $L_2(I)$ of the reminder of extension in Legendre polynomials. There are many results which can be used here. For more see also [6]. For example, using the standard results about Legendre and Jacobi polynomials we have

$$\inf_{w \in S} \|u - w\|_X \le C(p) q^{-p} \left[\int_{-1}^{1} (1-t^2)^p \left(\frac{d^p u}{dt^p}\right)^2 dt\right]^{1/2},$$

where an explicit form for $C(p)$ can be given. For example for $u = 1 - e^{-\lambda^2(t+1)}$ we get

$$\inf_{w\in S} \|u - w\|_X \le C_1(p) q^{-p} \lambda^{p-1}. \tag{2.28}$$

If u is an analytic function on I then we get exponential rate of convergence. Let us derive another estimate we mentioned above. Using Taylor's formula we get

$$\|u - w\|_X \le e^{2\lambda^2} \lambda^{2q} \frac{1}{q!}. \tag{2.29}$$

We see that for large q or small λ the rate is of order $(q!)^{-1}$.

2.7. Additional aspects.

We discussed the finite element method for the case when $S \subset X$, i.e., we were interested in the error measured in the L_2-norm. Because we can exchange the spaces S and V and still keeping the same properties of the bilinear form present, we can seek the finite element solution of the problem (2.7) also in the form

$$u(t) = \sum_{k=0}^{q} \alpha_k P_k(t)$$

$$u(-1) = a\lambda$$

and determine the coefficients α_k.

In our analysis we have assumed that the solution $u \in X$ and it has no trace and hence we cannot say anything about $u(+1)$. We can modify our approach so that value of $u(1)$ will be also the solution. To this end let

$$X^{[1]} = X \times \mathbb{R}^1, \quad \underline{u} = (u, \mu), \; u \in X, \; \mu \in \mathbb{R}^1$$

with the norm

$$\|u\|^2_{X^{[1]}} = \|u\|^2_X + |\mu|^2$$

and let $Y^{[1]}_\lambda$ be the space without the condition at $t = 1$ with the norm

$$\|v\|^2_{Y^{[1]}_\lambda} = \|- \frac{\dot{v}}{\lambda} + \lambda v\|^2_X + \left(\frac{v(1)}{\lambda}\right)^2$$

and the bilinear form will be

$$B(u,\mu;v) = \int_{-1}^{1} u\left(-\frac{\dot{v}}{\lambda} + \lambda v\right)dt + \frac{\mu v(1)}{\lambda}$$

which leads to a weak solution of the problem 2.7 with $\mu = u(1)$.

We discussed only the case of single interval. It is possible to treat also in an analogous way the case when the time interval is derived into subintervals, etc. We will elaborate on these or similar problems in the future.

3. The p-version for the parabolic problem

3.1. Preliminaries and problem formulation.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, Lipschitz domain with a piecewise analytic boundary Γ. Let further $D = I\times\Omega$, $I = (-1,1)$. Then we will consider the problem

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{on} \quad D \tag{3.1}$$

$$u = 0 \quad \text{on} \quad I\times\Gamma$$

$$u(-1,x) = g(x) \quad \text{on} \quad \Omega. \tag{3.2}$$

Let $X = X(D) = L_2(I,\mathring{H}^1(\Omega))$, where $\mathring{H}^1(\Omega)$ is the standard Sobolev space, as well as $H^0(\Omega) = L_2(\Omega)$ with

(3.3)
$$\|u\|^2_{\overset{\circ}{H}{}^1(\Omega)} = \int_\Omega |\nabla u|^2 dx$$

and

$$\|u\|^2_X = \int_{-1}^{1} \|u\|^2_{\overset{\circ}{H}{}^1(\Omega)} dt.$$

By $H^{-1}(\Omega)$ we denote the usual Sobolev space with the norm

(3.4)
$$\|u\|_{H^{-1}(\Omega)} = \sup_{v\in\overset{\circ}{H}{}^1(\Omega)} \frac{\left|\int_\Omega uvdx\right|}{\|v\|_{\overset{\circ}{H}{}^1(\Omega)}}.$$

By $\overset{\circ}{C}{}^{)}$ we denote the space

$$\overset{\circ}{C}{}^{)} = \{v \in C^\infty(\bar{D}) \mid v(t,x) \text{ has for any } t \in I \text{ compact support in } \Omega \text{ and } v(1,x) = 0\}.$$

We denote by $Y = Y(D)$ the completion of $\overset{\circ}{C}{}^{)}$ in the norm

$$\|v\|^2_Y = \int_{-1}^{1} (\|\dot{v}\|^2_{H^{-1}(\Omega)} + \|v\|^2_{\overset{\circ}{H}{}^1(\Omega)})dt,$$

where $\dot{v} = \frac{\partial v}{\partial t}$. We have

Lemma 3.1. Let $v \in \overset{\circ}{C}{}^{)}$ and $V(t,x)$ be such that for any $t \in I$, $V \in \overset{\circ}{H}{}^1(\Omega)$ and

(3.5)
$$\int_\Omega \nabla V(t,x)\nabla z dx = \int_\Omega vzdx \quad \forall\ z \in \overset{\circ}{H}{}^1(\Omega).$$

Then

$$\|v\|_{H^{-1}(\Omega)} = \|V\|_{\overset{\circ}{H}{}^1(\Omega)}.$$

Proof. Obviously $V(t,x)$ is uniquely determined by (3.5). We have then

$$\|v\|_{H^{-1}(\Omega)} = \sup_{z\in \mathring{H}^1(\Omega)} \frac{\left|\int_\Omega vzdx\right|}{\|z\|_{\mathring{H}^1(\Omega)}} = \sup_{z\in \mathring{H}^1(\Omega)} \frac{\left|\int_\Omega \nabla V\nabla zdx\right|}{\|z\|_{\mathring{H}^1(\Omega)}} = \|V\|_{\mathring{H}^1(\Omega)}. \quad \blacksquare$$

Hence we have

$$\|v\|_Y^2 = \int_{-1}^{1} (\|\dot{V}\|^2_{\mathring{H}^1(\Omega)} + \|v\|^2_{\mathring{H}^1(\Omega)})dt, \tag{3.6}$$

where V is defined by (3.5).

Let now (λ_j, u_j), $u_j \in \mathring{H}^1(\Omega)$ be the eigenpair of the eigenvalue problem

$$-\Delta u_j = \lambda_j^2 u_j \quad \text{on} \quad \Omega$$

$$u_j = 0 \quad \text{on} \quad \Gamma.$$

Then

$$\int_\Omega \nabla u_j \nabla zdx = \lambda_j^2 \int_\Omega u_j zdx \quad \forall\ z \in \mathring{H}^1(\Omega)$$

and any $u \in \mathring{H}^1(\Omega)$ can be written in the form

$$u = \sum_{i=1}^{\infty} \alpha_i u_i \tag{3.7}$$

with

$$\int_\Omega u^2 dx = \sum_{i=1}^{\infty} \alpha_i^2$$

$$\int_\Omega |\nabla u|^2 dx = \sum_{i=1}^{\infty} \alpha_i^2 \lambda_i^2.$$

Hence $X = X(D)$ is the set of the functions of the form

$$u(t,x) = \sum_{i=1}^{\infty} \alpha_i(t) u_i(x)$$

and

(3.8) $$\|u\|_X^2 = \int_{-1}^{1} \sum_{i=1}^{\infty} \alpha_i^2(t)\lambda_i^2 dt < +\infty.$$

Further, it is easy to see that if $v \in \mathring{C}^{)}$ then

(3.9a) $$v(t,x) = \sum_{i=1}^{\infty} \beta_i(t)u_i(x)$$

with

(3.9b) $$\beta_i(1) = 0$$

and

$$\|\dot{v}\|^2_{H^{-1}(\Omega)} = \int_{-1}^{1} \left[\sum_{i=1}^{\infty} \dot{\beta}_i^2(t)\frac{1}{\lambda_i^2}\right] dt.$$

Hence

(3.10) $$\|v\|_Y^2 = \int_{-1}^{1} \left[\sum_{i=1}^{\infty} (\dot{\beta}_i^2(t)\frac{1}{\lambda_i^2} + \beta_i^2(t)\lambda_i^2)\right] dt$$

and Y co[illegible]s of the set of functions of the form (3.9a,b) so that (3.10) is finite.

<u>Remark 3.1</u>. Because $\beta_i(t) \in H^1(I)$, $i = 1,2,\ldots$, $\beta_i(1)$ exists for $i = 1,2,\ldots$.

Using now Lemma 2.1, we see that the norms

(3.11) $$\|v\|_Y^2 = \int_{-1}^{1} \left[\sum_{i=1}^{\infty} (\dot{\beta}_i^2(t)\frac{1}{\lambda_i^2} + \beta_i^2(t)\lambda_i^2)\right] dt$$

and

(3.12) $$\|v\|_{\hat{Y}}^2 = \int_{-1}^{1} \left[\sum_{i=1}^{\infty} (-\dot{\beta}_i(t)\frac{1}{\lambda_i} + \beta_i(t)\lambda_i)^2\right] dt$$

are equivalent.

Let us define on $X \times Y$ the bilinear form

$$(3.13) \qquad B(u,v) = \int_{-1}^{1}\int_{\Omega}(-u\dot{v}+\nabla u\nabla v)\,dx\,dt = \int_{-1}^{1}\left[\sum_{i=1}^{\infty}(-\alpha_i\dot{\beta}_i + \lambda_i^2\alpha_i\beta_i)\right]dt.$$

(3.8) and (3.10) yield immediately that

$$(3.14) \qquad |B(u,v)| \le \|u\|_X\|v\|_Y$$

and using the same argument as in section 2.1 we get

$$(3.15a) \qquad \inf_{\substack{u\in X \\ \|u\|_X=1}} \sup_{\substack{v\in Y \\ \|v\|_Y\le 1}} |B(u,v)| \ge d_1 > 0$$

$$(3.15b) \qquad \inf_{\substack{v\in Y \\ \|v\|_Y=1}} \sup_{\substack{u\in X \\ \|u\|_X\le 1}} |B(u,v)| \ge d_1 > 0.$$

Let us now define problem $\mathcal{P}$. Given $F \in Y'$ find $u_0 \in X$ such hat

$$(3.16) \qquad B(u_0,v) = \mathbf{F}(v) \quad \forall\ v \in Y.$$

From (3.14) and (3.15) we have that problem $\mathcal{P}$ has unique solution for any $F \in Y'$ and

$$\|u_0\|_X \le C\|F\|_{Y'}$$

where C does not depend on F.

It is also easy to show that the solution u_0 of the problem $\mathcal{P}$ is a weak solution of (3.1) and (3.2) with

$$F(v) = \int_{\Omega} g(x)v(-1,x)\,dx + \int_{-1}^{1}\int_{\Omega} f(t,x)v(t,x)\,dx\,dt$$

in the sense that if u_0 is smooth and satisfies (3.1) and (3.2) then it satisfies also (3.16).

Remark 3.2. We restricted ourselves to the special case of $\Omega \subset \mathbb{R}^2$ with Lipschitz domain. We did it only for simplicity. For example in this section no restrictive assumption on Ω has been made at all.

3.2. The semidiscrete problem. Discretization in x.

In this section we will consider the semidiscrete (in t) solution of the problem $\mathcal{P}$. Let $R \subset \mathring{H}^1(\Omega)$ be a finite dimensional subspace of functions and

$$S = \{u \in X \mid u(t,x) \in R \quad \forall\, t \in I\}$$

$$V = \{v \in Y \mid v(t,x) \in R \quad \forall\, t \in I\}.$$

Let $u_s \in S$ be such that

$$B(u_s, v) = F(v) \quad \forall\, v \in V. \tag{3.17}$$

Then we will call u_s the semidiscrete solution of the problem $\mathcal{P}$. In (3.4) we have defined the norm $\|\cdot\|_{H^{-1}(\Omega)}$. Let $u \in R$, then we define

$$\|u\|^2_{H_R^{-1}(\Omega)} = \sup_{v \in R} \frac{\left|\int_\Omega uv\,dx\right|}{\|v\|_{\mathring{H}^1(\Omega)}}. \tag{3.18}$$

Obviously

$$\|u\|_{H_R^{-1}(\Omega)} \le \|u\|_{H^{-1}(\Omega)}. \tag{3.19}$$

We will say that the space R has property $\mathcal{K}$ if there is a number $\mathcal{H}(R) < 1$ such that

$$\|u\|_{H_R^{-1}(\Omega)} \ge \mathcal{H}(R)\|u\|_{H^{-1}(\Omega)} \tag{3.20}$$

holds for any $u \in R$.

Obviously $\mathcal{H}(R)$ depends on R. We will discuss it later.

Let now $\lambda_{R,i}, u_{R,i} \in R$ be the discrete eigenpair

$$\int_\Omega \nabla u_{R,i} \nabla z dx = \lambda_{R,i}^2 \int_\Omega u_{R,i} z dx \quad \forall\ z \in R.$$

Then if $u \in S$, $Q(R) = \dim R$

$$u(t,x) = \sum_{i=1}^{Q(R)} \tilde{\alpha}_i(t) u_{R,i}(x)$$

and

(3.21) $$\|u\|_X^2 = \int_{-1}^{1} \left[\sum_{i=1}^{Q(R)} \tilde{\alpha}_i^2(t) \lambda_{R,i}^2\right] dt.$$

Further, let $v \in V$

$$v(t,x) = \sum_{i=1}^{Q(R)} \tilde{\beta}_i(t) u_{R,i}(x)$$

and define

$$\|v\|_{Y_R}^2 = \int_{-1}^{1} \left[\sum_{i=1}^{Q(R)} (\dot{\tilde{\beta}}_i^2(t) \frac{1}{\lambda_{R,i}^2} + \tilde{\beta}_i^2(t) \lambda_{R,i}^2)\right] dt = \int_{-1}^{1} \left(\|\dot{v}\|_{H_R^{-1}(\Omega)}^2 + \|v\|_{\mathring{H}^1(\Omega)}^2\right) dt.$$

Because of (3.19) we have

(3.22) $$\|v\|_{Y_R} \le \|v\|_Y$$

and if R has property $\mathcal{K}$ then

(3.23) $$\mathcal{K}(R) \|v\|_Y \le \|v\|_{Y_R}.$$

Analogously as in Section 3.1 we define

$$\|v\|_{\hat{Y}_R}^2 = \int_{-1}^{1} \left[\sum_{i=1}^{\infty} (-\dot{\tilde{\beta}}_i(t) \frac{1}{\lambda_{R,i}} + \tilde{\beta}_i(t) \lambda_{R,i})^2\right] dt$$

then the norms $\|\cdot\|_{Y_R}$ and $\|\cdot\|_{\hat{Y}_R}$ are equivalent.

<u>Theorem 3.2</u>. Let R have property $\mathcal{K}$, $u \in S$, $v \in V$. Then

i)

$$|B(u,v)| \le \|u\|_X \|v\|_Y, \tag{3.24}$$

ii)

$$\inf_{\substack{u\in S\\ \|u\|_X=1}} \sup_{\substack{v\in V\\ \|v\|_Y\le 1}} |B(u,v)| \ge C_2 \mathcal{H}(R), \tag{3.25}$$

iii)

$$\sup_{\substack{u\in S\\ \|u\|_X=1}} |B(u,v)| > 0 \quad \forall\, v \in V,\ v \ne 0, \tag{3.26}$$

where C_1 and C_2 are independent of R.

<u>Proof</u>. (3.24) follows from (3.14) and (3.22). Let $u \in S$, $u = \sum_{i=1}^{Q(R)} \tilde{\alpha}_i u_{R,i}$, $v \in V$, $v = \sum_{i=1}^{Q(R)} \tilde{\beta}_i u_{R,i}$. Then

$$B(u,v) = \int_{-1}^{1} \left(\sum_{i=1}^{Q(R)} \lambda_{R,i} \tilde{\alpha}_i \left(- \frac{\dot{\tilde{\beta}}_i}{\lambda_{R,i}} + \lambda_{R,i} \tilde{\beta}_i \right) \right) dt.$$

Using the results of section 2, given $u_0 \in S$ there exists $v \in V$ such that

$$B(u,v) = \|u\|_X^2$$

and using equivalency of the norms Y_R and $\hat{Y}_R$ we get

$$\frac{B(u,v)}{\|v\|_Y} = \|u\|_X \frac{\|u\|_X}{\|v\|_{Y_R}} \frac{\|v\|_{Y_R}}{\|v\|_Y} \ge C \|u\|_X \mathcal{H}(R)$$

which leads immediately to (3.25). (3.26) follows also easily as in Section 2. ■

Using now Theorem 6.2.1 of [4] we get immediately

<u>Theorem 3.3</u>. Let R have property K, u_0 and u_S are the solutions of (3.16) and (3.17), respectively. Then

$$\|u_0 - u_s\|_X \le \frac{C}{\mathcal{K}(R)} \inf_{w \in S} \|u_0 - w\|_X$$

and C is independent of R. ■

Let us now discuss the accuracy of u_s. We have

$$B(u_0 - u_s, v) = 0 \quad \forall\, v \in V.$$

Write $u_s = \rho + \xi$, $\rho, \xi \in S$, where ρ will be judiciously selected. Then we get

$$B(\xi, v) = B(u_0 - \rho, v) \quad \forall\, v \in V$$

$$B(u_0 - \rho, v) = \int_{-1}^{1} \int_{\Omega} (-(u_0 - \rho)\dot{v} + \nabla(u_0 - \rho)\nabla v)\,dx\,dt.$$

Let us select $\rho(x,t) = P_0 u_0(x,t)$, where P_0 is L_2-orthogonal projection of $\mathring{H}^1(\Omega)$ onto R (for every fixed t). Then for any $v \in V$

$$B(u_0 - \rho, v) = \int_{-1}^{1} \int_{\Omega} (\nabla(u_0 - P_0 u_0)\nabla v)\,dx\,dt.$$

The bilinear form $B(u,v)$ satisfies the conditions (3.24), (3.25), and (3.26) without the property $\mathcal{K}$ if using the norm Y_R instead of Y. Because

$$\int_{-1}^{1} \int_{\Omega} (\nabla(u_0 - P_0 u_0)\nabla v)\,dx\,dt \le \|u_0 - P_0 u_0\|_X \|v\|_{Y_R}$$

we get

$$\|\xi\|_X \le C\|u_0 - P_0u_0\|_X,$$

where C is independent of R. Hence we have

<u>Theorem 3.4</u>. Let u_0 be the solution of the problem $\mathcal{P}$. Then

$$\|u_s - u_0\|_X \le C\|u_0 - P_0u_0\|_X,$$

where by P_0 we denoted the L_2-orthogonal projection of X onto R without assuming the property $\mathcal{K}$. □

3.3. <u>The condition</u> $\mathcal{K}$ <u>and related results</u>.

In the previous section we mentioned condition $\mathcal{K}$ of the subspace R. We will now elaborate more on it. Let

$$\xi(R) = \sup_{u\in R} \frac{\|u\|_{\mathring{H}^1(\Omega)}}{\|u\|_{L_2(\Omega)}} < +\infty \tag{3.27}$$

$$\eta(R) = \sup_{\substack{u\in\mathring{H}^1(\Omega) \\ \|u\|_{\mathring{H}^1(\Omega)}\le 1}} \|u - P_1u\|_{L_2(\Omega)} < +\infty \tag{3.28}$$

$$\lambda(R) = \sup_{\substack{u\in\mathring{H}^1(\Omega) \\ \|u\|_{\mathring{H}^1(\Omega)}\le 1}} \|u - P_0u\|_{L_2(\Omega)} < +\infty. \tag{3.29}$$

ξ, η, and λ depend on R. Here we denoted P_0, resp. P_1, the L_2-, resp. H^1-projection operator of $\mathring{H}^1(\Omega)$ on R. We will call now R to be (ξ,η,λ)-regular.

<u>Theorem 3.5</u>. Assume that R is (ξ,η,λ)-regular. Then R has property $\mathcal{K}$ and

$$\mathcal{H}(R) \ge (1+\xi\eta)^{-1}.$$

Proof. Let $u \in R$. Then

$$\|u\|_{H^{-1}(\Omega)} = \sup_{v\in \mathring{H}^1(\Omega)} \frac{|\int_\Omega uvdx|}{\|v\|_{\mathring{H}^1(\Omega)}}$$

and

$$\|u\|_{H_R^{-1}(\Omega)} = \sup_{v\in R} \frac{|\int_\Omega uvdx|}{\|v\|_{\mathring{H}^1(\Omega)}}.$$

Hence, we have

$$\|u\|_{H^{-1}(\Omega)} = \sup_{v\in \mathring{H}^1(\Omega)} \frac{|\int_\Omega uP_1vdx|}{\|v\|_{\mathring{H}^1(\Omega)}} + \sup_{v\in \mathring{H}^1(\Omega)} \frac{|\int_\Omega u(v-P_1v)dx|}{\|v\|_{\mathring{H}^1(\Omega)}}$$

$$\le \|u\|_{H_R^{-1}(\Omega)} + \eta(R)\|u\|_{L_2(\Omega)}.$$

Next

$$\|u\|_{H_R^{-1}(\Omega)} \ge \sup_{v\in R} \frac{|\int_\Omega uvdx|}{\xi(R)\|v\|_{L_2(\Omega)}} = \frac{\|u\|_{L_2(\Omega)}}{\xi(R)}.$$

Hence

$$\frac{\|u\|_{H^{-1}(\Omega)}}{\|u\|_{H_R^{-1}(\Omega)}} \le 1 + \xi\eta. \qquad \blacksquare$$

Theorem 3.6. Assume that R is (ξ,η,λ)-regular. Then for any $u \in \mathring{H}^1(\Omega)$, $\|u\|_{\mathring{H}^1(\Omega)} = 1$,

$$\|u - P_0u\|_{\mathring{H}^1(\Omega)} \le (1+\eta(R)+\lambda(R))\xi(R).$$

Proof. We can write

$$u - P_0u = u - P_1u + P_1u - P_0u.$$

Hence

$$\|u - P_0 u\|_{\mathring{H}^1(\Omega)} \le \|u - P_1 u\|_{\mathring{H}^1(\Omega)} + (\eta(R)+\lambda(R))\xi(R) \le (1+\eta(R)+\lambda(R))\xi(R). \qquad \blacksquare$$

Remark 3.3. In the above theorem we do not need to use operator P_1. It is enough to use any other one $\tilde{P}_1$ with $\tilde{\eta}(R)$ instead to $\eta(R)$. Then Theorem 3.5 holds when $\eta(R)$ is replaced by $\tilde{\eta}(R)$. For example we can use some type of an interpolation operator which in the case of unsmooth domain Ω yields $\tilde{\eta}(R) < \eta(R)$.

The conditions (3.27) - (3.29) are closely related to the inverse property and the duality principle in the theory of finite element method. Let us mention here some known results.

i) Consider finite element method with a quasi-uniform mesh. Then

$$\xi(R) \le h^{-1}C(p),$$

where h is the size of the element and C depends on p and the mesh but is independent of h.

In the case of the p-version we have

$$\xi(R) \le p^2 Q$$

where Q depends on the mesh.

ii) Assuming that Ω is smooth, or all its angles are $\le \pi$, we have

$$\eta(R) \le C(p)h$$

or

$$\eta(R) \le p^{-1}Q.$$

iii) Finally without any additional assumption on the mesh we get

$$\lambda(R) \le Ch$$

$$\lambda(R) \le Cp^{-1}.$$

Remark 3.4. In this section we did not use any assumptions about Ω and its dimension, except in the last part, where we listed the results about the finite element space R.

Let us now analyze the p-version for the one-dimensional (in x) case. Assume now that

$$R(p) = \{u \in \mathring{H}^1(I) \mid u \text{ is a polynomial of degree } p\}.$$

Then we have (see e.g. [6])

$$\xi(R) \le Cp^2$$

and the factor p^2 is the best possible. Further we have

$$\|u - P_1 v\|_{L_2(I)} \le Cp^{-1}\|v\|_{\mathring{H}^1(I)}.$$

Hence we have

$$\eta(R) \le Cp^{-1}$$

and

(3.30) $$\mathcal{H}(R) \ge Cp^{-1}.$$

Similarly we get

(3.31) $$\|u-P_0 u\|_{\mathring{H}^1(I)} \le Cp\|u\|_{\mathring{H}^1(I)}.$$

Let us show now that (3.31) can be improved.

Lemma 3.7. Let $s \ge 0$ integer, $u \in \mathring{H}^1(I)$ and

$$\int_{-1}^{1} (u^{[s+1]})^2 (1-x^2)^s dx = A < +\infty,$$

where

$$u^{[s+1]} = \frac{d^{s+1}u}{dx^{s+1}}.$$

Then

$$\|u - P_0 u\|_{\mathring{H}^1(I)} \le C(s)p^{-s+(1/2)}A^{1/2}. \tag{3.32}$$

Proof. We can assume that u is smooth because of density of smooth functions. Assume that

$$u = \sum_{m=1}^{\infty} a_m(P_{m+1} - P_{m-1})$$

where P_m are as before Legendre polynomials. Then

$$u' = \sum_{m=1}^{\infty} a_m(2m+1)P_m$$

and

$$P_0 u = \sum_{m=1}^{p-1} b_m(P_{m+1} - P_{m-1}).$$

For the sake of simplicity we will restrict ourselves to the case when u is symmetric. The antisymmetric case can be treated similarly. In our symmetric case we have $b_m = a_m = 0$ for all m even. Let $m = 2k-1$, $k = 1,2,\dots,k_0$, $k_0 = \frac{p}{2}$, p even. Let us assume that $b_m = a_m + z_m$. Because P_0 is L_2-projection we have

$$\int_{-1}^{1}\left[\sum_{k=1}^{k_0} z_{2k-1}(P_{2k} - P_{2k-2}) - a_{2k_0+1}(P_{2k_0+2} - P_{2k_0})\right](P_{i+1} - P_{i-1})dx = 0$$

$$i = 1,2,\cdots,p-1.$$

This system leads to the following system of linear equations:

$$(2+\tfrac{2}{5})z_1 - \tfrac{2}{5}z_3 = 0 \quad (i = 1)$$

$$-\frac{2}{5}+\left(\frac{2}{5}+\frac{2}{9}\right)z_3-\frac{2}{9}z_5 = 0 \quad (i = 3)$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$-\frac{2}{4k-3}\,z_{2(k+1)-1}+\left(\frac{2}{4k-3}+\frac{2}{4k+1}\right)z_{2k+1}-\frac{2}{4k+1}z_{2(k+1)-1} = 0 \quad (i = 2k-1)$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\frac{2}{4k_0-3}z_{2k_0-3}+\left(\frac{2}{4k_0-3}+\frac{2}{4k_0+1}\right)z_{2k_0-1} = -\frac{2}{4k_0+1}a_{2k_0+1} \quad (i = 2k_0-1).$$

This system can be solved explicitly to obtain

$$z_{2k-1} = -C_{k_0}k(2k-1)a_{2k_0+1}, \quad k = 1,\dots,k_0, \quad 2k_0+1 = p+1$$

in which

$$C_{k_0} = \frac{4k_0-3}{8k_0^3+6k_0^2-5k_0-3}.$$

Hence we have

$$\|u' - (P_0u)'\|^2_{L_2(I)} = \sum_{k=1}^{k_0} z^2_{2k-1}(4k-1) + \sum_{k=k_0+1}^{\infty} a^2_{2k-1}(4k-1) \le Ca^2_{2k_0+1}k_0^2 + Q,$$

where $Q = \sum_{k=k_0+1}^{\infty} a^2_{2k-1}(4k-1)$ and C is bounded from above and below independently of k_0. Because of our assumption we have

$$\int_{-1}^{1}(u')^2dx = \sum_{m=1}^{\infty} a_m^2(2m+1)$$

and

$$\sum_{m=1}^{\infty} a_m^2(2m+1)^{1+2s} \le C\int_{-1}^{1}(u^{[s+1]})^2(1-x^2)^s dx \le CA.$$

Hence

$$(a_{2k_0+1})^2 \le CA(2k_0+1)^{-1-2s}$$

and

$$\sum_{k=k_0+1}^{\infty} a_{2k-1}^2(4k-1) \le Ck_0^{-2s}.$$

Therefore

$$\|u' - (P_0u)'\|_{L_2(I)}^2 \le CA(k_0^{1-2s}+k_0^{-2s})$$

which yields (3.32). ∎

By a similar procedure it is possible to show also that $\mathcal{H}(R) \ge p^{-1/2}$.

Remark 3.5. In Lemma 3.7, we have assumed that s is an integer. We can generalize the results by interpolation of weighted spaces, (see, e.g. [7]) for s nonintegers.

3.4. Numerical examples.

Consider now the problem (3.1), (3.2) with $\Omega = I$, $f = 0$ and

i) $g_1(x) = 1 - x^2$

ii) $g_2(x) = 1$

iii) $g_3(x) = \cos\frac{\pi x}{2}$.

Denoting $u_1(t,x)$ the solution of our problem with the initial function g_1 we can extend it antisymmetrically with respect to $x = \pm 1$ into a periodic function defined on $I\times\mathbb{R}$. We will denote the extended function once more by u_1. Let now $v = \frac{\partial^3 u_1}{\partial x^3}$. Then v satisfies the equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \quad \text{on} \quad \mathbb{R}\times I$$

and

$$v(-1,x) = \sum_{k=-\infty}^{+\infty} (-1)^k \vartheta(x-(2k+1))$$

where ϑ is the Dirac function. Hence,

$$v(t,x) = C \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{t}} (-1)^k e^{-\frac{(x-(2k+1))^2}{t}}$$

and we get

$$\int_{-1}^{1} \left[\frac{\partial^n v}{\partial x^n} (t,x)\right]^2 (1-x^2)^{2+n} dx \approx t^{(-n/2)+(1/2)}$$

and hence

(3.33) $$\|u_1 - P_0 u_1\|^2_{\mathring{H}^1(\Omega)} \le Cp^{-(3/2+n)2} t^{(-n/2)+(1/2)}.$$

By interpolation theory (3.33) holds not only for n integers but also for n real. Hence we can choose $n = 3 - \varepsilon$ and get

(3.34) $$\|u_1 - P_0 u_1\|_X \le C(\varepsilon) p^{-(4.5-\varepsilon)}.$$

Similarly we get

$$\|u_2 - P_0 u_2\|_{\mathring{H}^1(\Omega)} \le p^{-(n-1/2)2} t^{(-n/2)-(1/2)}$$

and

(3.35) $$\|u_2 - P_0 u_2\|_X \le C(\varepsilon) p^{-(0.5-\varepsilon)},$$

where in (3.34) and (3.35) $\varepsilon > 0$ is arbitrary. In case iii) we obviously have

$$u_3(t,x) = \cos \frac{\pi x}{2} e^{-\frac{\pi^2}{2} t}$$

and hence

$$\|u_3 - P_0 u_3\|_{\mathring{H}^1(\Omega)} \le C(k) p^{-k}$$

for arbitrary $k > 0$. Using the results from [6] we get

$$\|u_3 - P_0 u_3\|_X \le Cr^p \tag{3.36}$$

with $r < 1$ and C which can be estimated.

Applying ow theorem 3.4 together with the estimates (3.34), (3.35), and (3.36) we get

$$\|u_{1,s} - u_1\|_X \le Cp^{-(4.5-\varepsilon)} \tag{3.37a}$$

$$\|u_{2,s} - u_2\|_X \le Cp^{-(0.5-\varepsilon)} \tag{3.37b}$$

$$\|u_{3,s} - u_3\|_X \le Cr^p \tag{3.37c}$$

with $\varepsilon > 0$ arbitrary and $r < 1$.

Figure 3.1 shows the computed error of $\|u_{i,s} - u_i\|_X$ in the log log scale. Figure 3.1 also shows the slopes indicated by our estiamtes. We can see a very good agreement for u_1 and u_3. In the case u_2 the numerical experiment seems to indicate a better result (rate p^{-1} and not $p^{-1/2}$) in the computed range.

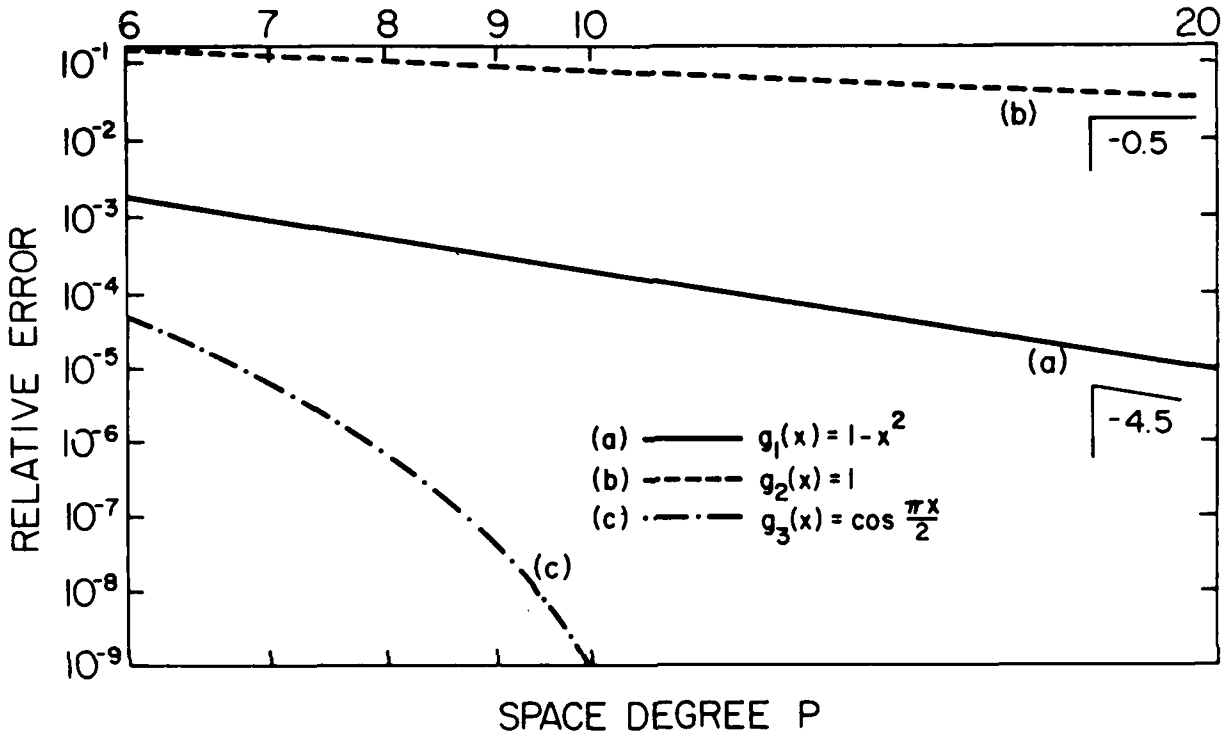

Figure 3.1. The relative error for the semidiscrete method (discretization in x) with the initial functions:

(a) $g_1(x) = 1 - x^2$, slope based on (3.37a);

(b) $g_2(x) = 1$, slope based on (3.37b);

(c) $g_3(x) = \cos \frac{\pi x}{2}$.

3.5. <u>The semidiscrete problem: Discretization in t.</u>

In this section we will consider the case when we discretize the variable t, while keeping x continuous. This is essentially the p-version of the Rothe method (see [8]). Let

$$T_{q-1}(I) = \{u \in L^2(I) \mid u \text{ is a polynomial of degree } q-1\}$$

$\mathring{T}_q^{)}(I) = \{v \in L^2(I) \mid v$ is a polynomial of degree q and $v(1) = 0\}$.

Define now

$$S_q = T_{q-1} \times \mathring{H}^1(\Omega)$$

$$V_q = \mathring{T}_q^{)} \times \mathring{H}^1(\Omega).$$

Obviously $S_q \subset X$ and $V_q \subset Y$, where X and Y are the spaces introduced in section 3.1. For any $u \in S_q$, resp. $v \in V_q$, we have

$$u(t,x) = \sum_{i=1}^{\infty} \alpha_i(t) u_i(x)$$

where u_i are the eigenfunctions as in section 3.1, with

$$\alpha_i(t) \in T_{q-1}$$

and

$$v = \sum_{i=1}^{\infty} \beta_i(t) u_i(x)$$

with

$$\beta_i(t) \in \mathring{T}_q^{)}.$$

The norms $\|u\|_X$ and $\|v\|_Y$ are now given by (3.8) and (3.11). Defining the bilinear forms $B(u,v)$ on $S_q \times V_q$ by (3.13) we see that (3.14) holds. Using now Theorem 2.3 we get

$$\inf_{\substack{v \in V_q \\ \|v\|_Y = 1}} \sup_{\substack{u \in S_q \\ \|u\|_X \le 1}} |B(u,v)| \ge Cq^{-1/2}$$

$$\inf_{\substack{u \in S_q \\ \|u\|_X = 1}} \sup_{\substack{v \in V_q \\ \|v\|_Y \le 1}} |B(u,v)| \ge Cq^{-1/2}$$

Hence we have

Theorem 3.8. Let u_0 be the solution of the problem $\mathcal{P}$ and u_s its semi-discrete solution (discretization in t). Then

$$\|u_s - u_0\|_X \le Cq^{1/2} \inf_{w \in S_q} \|u - w\|_X \tag{3.38}$$

Theorem 3.9. Assume that $f = 0$ in the problem $\mathcal{P}$ and that the conjecture A holds. Then

$$\|u_s - u_0\|_X \le C\varphi(q) \inf_{w \in S_q} \|u - w\|_X, \tag{3.39}$$

where $\varphi = 1$ or $\varphi(q) = \log(q)$.

Let us remark that there are solutions of the problem $\mathcal{P}$ with f being rough so that the factor $q^{1/2}$ in (3.38) is present. On the contrast if f is smooth then (3.39) is applicable. Let us now consider the examples discussed in section 3.4. Obviously we can write

$$u(t,x) = \sum_{k=1}^{\infty} a_k e^{-(\frac{2k-1}{2})^2 \pi^2 (t+1)} \cos(2k-1)\frac{\pi x}{2}$$

where

$$|a_k| \le ck^{-3} \quad \text{for the function } u_1$$

$$|a_k| \le ck^{-1} \quad \text{for the function } u_2$$

and

$$a_k = 0 \quad k = 2,3,\dots \quad \text{for the function } u_3.$$

Using now (2.28) we get

$$\inf_{w \in S_q} \|u - w\|_X^2 \le \sum_{k=1}^{\infty} k^2 a_k^2 (k\pi)^{2n-2} q^{-2n}$$

and hence we can use for u_1, $n = 3 - \varepsilon$ and get

(3.40a) $$\|u_{1,s} - u_1\|_X \le Cq^{-(2-\varepsilon)}$$

analogously

(3.40b) $$\|u_{2,s} - u_2\|_X \le Cq^{\varepsilon}$$

or possibly $q^{-2.5+\varepsilon}$ and $q^{-0.5+\varepsilon}$ if conjecture A holds and

(3.40c) $$\|u_{3,s} - u_3\|_X \le Cq^{-k}$$

for k arbitrarily large.

Figure 3.2 presents the computed error of $\|u_{i,s} - u_i\|_X$ in the log log scale. Figure 3.2 also shows the slope indicated by our estimates taking into account our conjecture A. We can see a very good agreement for all cases.

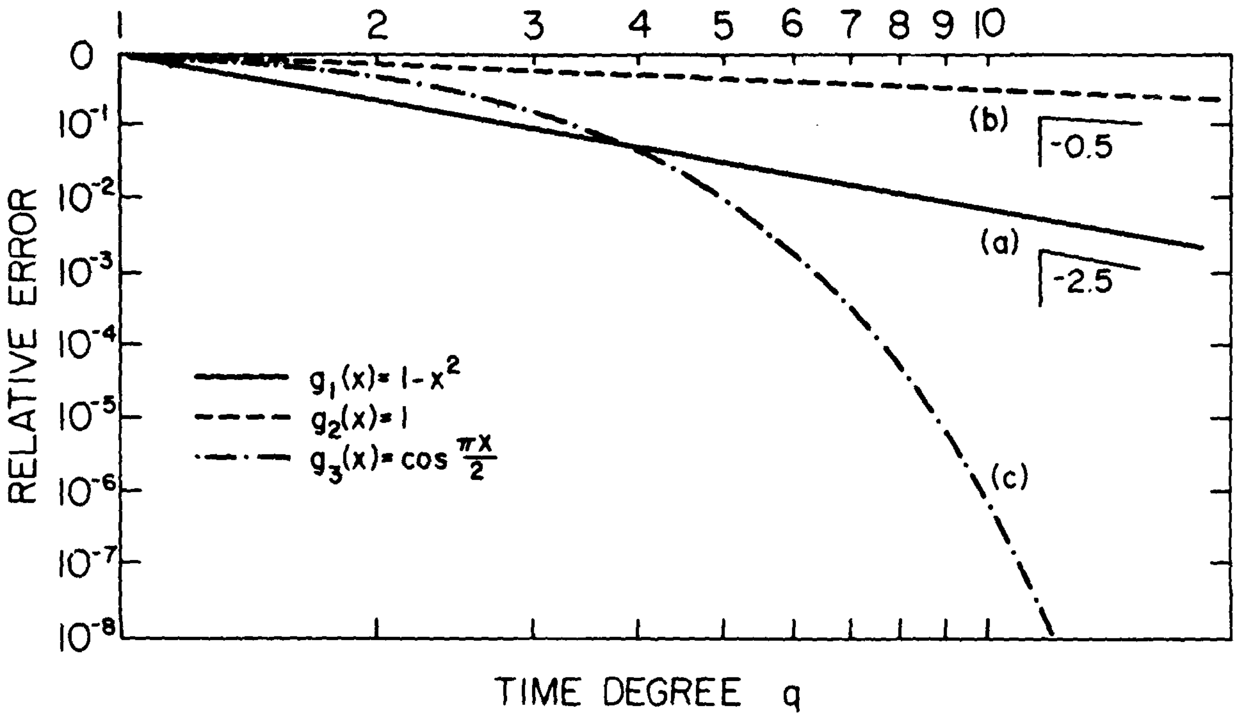

Figure 3.2. The relative error for the semidiscrete method (discretization in t) with the initial functions:

(a) $g_1(x) = 1 - x^2$, slope based on (3.40a) with conjecture A;

(b) $g_2(x) = 1$, slope based on (3.40b) with conjecture A;

(c) $g_3(x) = \cos\frac{\pi x}{2}$.

3.5. <u>The complete discretization</u>.

In this section we will consider the general case when the discretization in x and t has been made simultaneously. Let us define

$$S = S(R,q) = T_{q-1} \times R$$

$$V = V(R,q) = \mathring{T}_q^{)} \times R.$$

Now we obviusly have for $u \in S$, $v \in V$

(3.41a) $$|B(u,v)| \le \|u\|_X \|v\|_Y$$

and combining the results of previous sections we get

(3.41b) $$\inf_{\substack{u \in S \\ \|u\|_X \le 1}} \sup_{\substack{v \in V \\ \|v\|_Y = 1}} |B(u,v)| \ge C\mathcal{H}(R)q^{-1/2}$$

(3.41c) $$\inf_{\substack{v \in V \\ \|v\|_Y = 1}} \sup_{\substack{u \in S \\ \|u\|_X \le 1}} |B(u,v)| \ge C\mathcal{H}(R)q^{-1/2}$$

where C is independent of R and q. Hence we get

<u>Theorem 3.10</u>. Let u_0 be the solution of the problem $\mathcal{P}$ and $u_s \in S$ be the finite element solution

$$B(u_s, v) = F(v) \quad \forall\, v \in V.$$

Then

$$\|u_s - u_0\|_X \le Cq^{1/2}(\mathcal{H}(R))^{-1} \inf_{w \in S} \|u_0 - w\|_S.$$

In the case that $f = 0$ we get

$$\|u_s - u_0\|_X \le C\varphi(q)(\mathcal{H}(R))^{-1} \inf_{w \in S} \|u_0 - w\|_S$$

where $\varphi(q) < 1$ or $\varphi(q) < \log(q)$ when the conjecture A holds.

Let us now proceed for another estimate. Denote μ_T solution when the variable t is discretized, while x is continuous. Let $\|\mu_T - u_0\|_X \le \varepsilon_1$. Proceeding as before we get now

$$\|u_s - u_0\|_X^2 \le C\left[\varepsilon_1^2 + \int_{-1}^{1} \|\mu_T - P_0\mu_T\|^2_{\mathring{H}^1(\Omega)} dt\right]$$

and

$$\mu_T - P_0\mu_T = (\mu_T - u_0) - P_0(\mu_T - u_0) + u_0 - P_0u_0.$$

Using now Theorem 3.6, we get

$$\int_{-1}^{1} \|\mu_T - P_0\mu_T\|^2_{\mathring{H}^1(\Omega)} dt \le C \int_{-1}^{1} \|u_0 - P_0u_0\|^2 dt + \varepsilon_1^2\Big[(1+\eta(R)+\lambda(R))\xi(R)\Big]^2$$

and hence

$$\|u_s - u_0\|_X \le C(\varepsilon_1(1+\eta(R)+\lambda(R))\xi(R)+\varepsilon_2)$$

where ε_1 is the error of the semidiscrete method (the discretization of t), which was discussed in section 3.5 and ε_2 is the error of the semidiscrete method (the discretization in x) which was discussed in section 3.2.

Let us consider now the numerical example we discussed earlier. In this case we have

$$(1+\eta(R)+\lambda(R))\xi(R) \le Cp$$

but using Lemma 3.7 we can replace this term by $Cp^{1/2}$. hence we get

(3.42a) $$\|u_1 - u_{1,s}\|_X \le Cp^{1/2}(p^{-5+\varepsilon}+q^{-2.5+\varepsilon})$$

(3.42b) $$\|u_2 - u_{2,s}\|_X \le Cp^{1/2}(p^{-1+\varepsilon}+q^{-0.5+\varepsilon})$$

(3.42c) $$\|u_3 - u_{3,s}\|_X \le Cp^{1/2}(p^{-k}+q^{-k})$$

where $\varepsilon > 0$, $k < 0$ arbitrary

The rates in q are increased by 1/2 since the conjecture A has been used. The estimates (3.42a,b,c) indicate the optimal combination in q and p. Taking into account the fact that the computational work (i.e., number of arithmetic operations) is proportional to qp^3, we can see that the optimal choices are $q \sim p^2$ in the first and second cases, and $q \sim p$ in the last case. Using the estimate (3.42a,b,c) we get the optimal relationship between the error and the work:

(3.43a) $$E = CW^{-0.9} \quad \text{for} \quad g(x) = 1 - x^2$$

(3.43b) $$E = CW^{-0.1} \quad \text{for} \quad g(x) = 1$$

(3.43c) $$E = CW^{-k}, \quad k > 0 \ \text{arbitrary} \quad \text{for} \quad g(x) = \cos\frac{\pi x}{2}$$

Figures 3.3 - 3.5 present the performance of the complete discretization via p-version of the finite element method. Figures show the slopes indicated by the functions given by (3.43a) and (3.43b). We can see a very good agreement for u_2 (Figure 3.4, numerical rate of convergence is slightly better than the theoretical one) and for u_3 (Figure 3.5). In the case (a), i.e., for $g_1(x) = 1 - x^2$ (Figure 3.3) the numerical rate of convergence is very close to the theoretical one, but the best result is obtained for the combination $p = [q^{1.8}]$ instead of $p = q^2$.

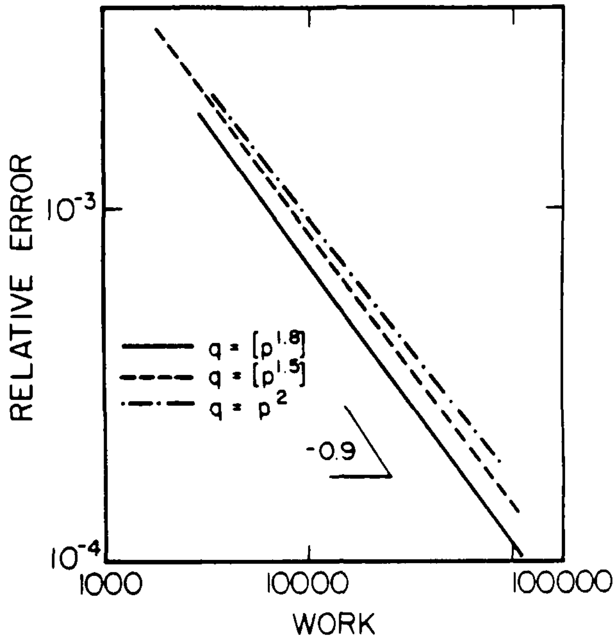

Figure 3.3. The performance of the complete discretization via p-version of the finite element method for various combinations between the space degree p and the time degree q. The initial function is $g_1(x) = 1 - x^2$. The slope is based on (3.43a).

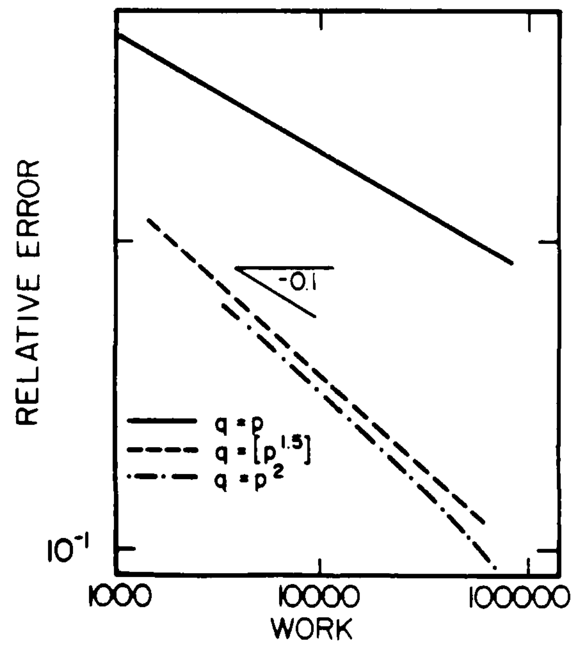

Figure 3.4. The performance of the complete discretization via p-version of the finite element method for various combinations between the p and q. The initial function is $g_2(x) = 1$. The slope is based on (3.43b).

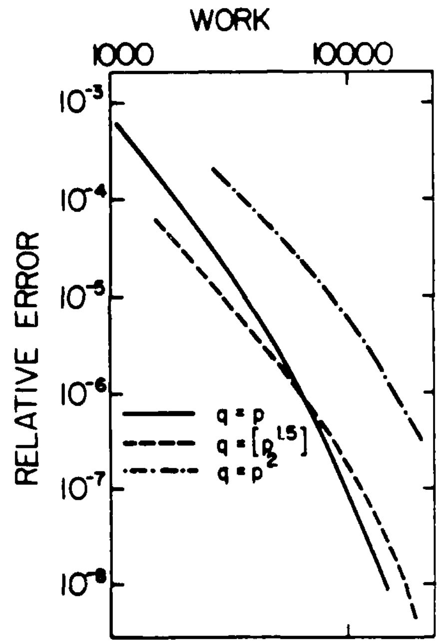

Figure 3.5. The performance of the complete discretization via p-version of the finite element method for various combinations between p and q. The initial function is $g_3(x) = \cos\frac{\pi x}{2}$.

3.7. <u>Additional comments</u>.

In section 2.6 we mentioned that the theory can be extended where composite intervals are used. This, of course, can be applied here too. For example the estimate (3.42a) indicates that the refinement in the variable t would be advantageous.

We have assumed that the discretization in x is the same for all $t \in I$. If I is partitioned into I_j then also discretization in x can be different for various I_j. Then we get the h-p version for parabolic equation. These and analogous questions will be discussed in forthcoming papers.

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